Measure Theoretic Weighted Model Integration

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Abstract

Weighted model counting (WMC) is a popular framework to perform probabilistic inference with discrete random variables. Recently, WMC has been extended to weighted model integration (WMI) in order to additionally handle continuous variables. At their core, WMI problems consist of computing integrals and sums over weighted logical formulas. From a theoretical standpoint, WMI has been formulated by patching the sum over weighted formulas, which is already present in WMC, with Riemann integration. A more principled approach to integration, which is rooted in measure theory, is Lebesgue integration. Lebesgue integration allows one to treat discrete and continuous variables on equal footing in a principled fashion. We propose a theoretically sound measure theoretic formulation of weighted model integration, which naturally reduces to weighted model counting in the absence of continuous variables. Instead of regarding weighted model integration as an extension of weighted model counting, WMC emerges as a special case of WMI in our formulation.

Keywords: weighted model counting, weighted model integration, measure theory, probabilistic inference

1. Introduction

Weighted model counting (WMC) [1], in combination with knowledge compilation [2], has emerged as the go-to technique to perform inference in probabilistic graphical models [3] and probabilistic programming languages [4] with discrete random variables. A major drawback of standard WMC, however, is its limitation to discrete (random) variables and hence to discrete probability distributions and weight functions only. This puts considerable restrictions on the problems that can be modeled. Weighted model integration (WMI) [5] is a recent extension of the WMC formalism that tackles this deficiency and allows additionally for continuous variables.

Example 1. Consider the example of the WMI problem in Figure 1. The problem has two continuous random variables (x and y) and one Boolean random variable. The Boolean random variable determines which weight function is chosen ($2x+y^2$ or $x^3+y/3$). For instance, if we have a fair Boolean random variable both weight functions have a probability of $1/2$ each to be selected. Given the outcome of the Boolean variable, constraints on the real variables then produce the feasible regions (the red region and the two blue regions) – inside of which the weight functions remain non-zero. WMI tackles the problem of computing the integral over the feasible regions.

Since the inception of WMI, a plethora of inference algorithms have emerged following the WMI paradigm. Some of which perform exact inference [6, 7, 8, 9, 10, 11, 12, 13], or

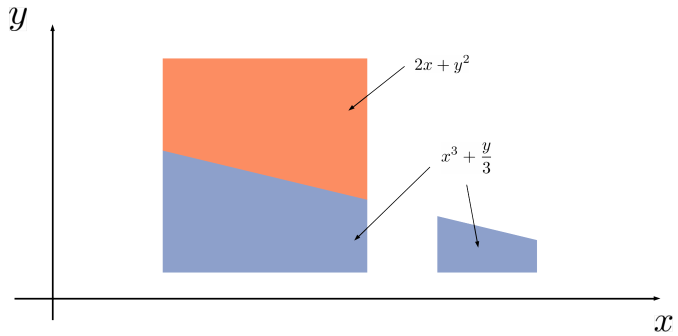

Figure 1: Geometric representation of a WMI problem.

approximate inference [14, 11], or solve a subclass of WMI problems efficiently [15, 16, 17] — demonstrating an avid interest from the research community. The pywmi toolbox for WMI solvers [18] reassembles some of these efforts in a single Python library. We refer the reader to [19] for a recent survey on WMI.

All of the above cited works formalize WMI as a combination of Riemann integration and summation and none leverages the power of Lebesgue integration in order to formally define weighted model integration. This is rather astounding as Lebesgue integration is a natural fit to formalize integration/summation in such a discrete-continuous setting. We speculate this is due to the technical overhead involved with Lebesgue integration compared to Riemann integration. In this paper we show how the problem of weighted model integration can be defined in terms of Lebesgue integration and place WMI in a measure theoretic setting. We hope this work will help bridge theoretical distinctions between different approaches to WMI and create a unified and cohesive view on probabilistic inference problems in Boolean, discrete, and continuous domains.

Traditionally, probability theory has been one of the main domains of application of Lebesgue integration and measure theory. Probability is in fact naturally represented as special type of measure function. Given that WMI is inseparably connected to probabilistic inference, it is convenient to formalize WMI in a measure theoretic setting. Effectively, this extends the currently used class of Riemann integrable WMI problems [5] to the class of Lebesgue integrable problems.

The remainder of the paper is organized as follows. Appendix A discusses the necessary background on measure theory. In Section 2, the problems of WMC and WMI are introduced (using the current formulation based on summation and Riemann integration). Section 3, the central part of the paper, first presents the formalization of WMC in a measure theoretic setting (Subsection 3.2), followed by an analogous treatment of WMI (Subsection 3.3). Succeedingly, Section 4 deals with the important category of probabilistic weight functions, which can directly be represented as a measure, again first in the Boolean setting (Subsection 4.1), and then in the hybrid one (Subsection 4.2). In Section 5, we then demonstrate the usefulness of measure theoretic WMI for discussing computational complexity considerations of WMI problems, thereby generalizing prior work [16, 17, 20]. We end the paper with concluding remarks in Section 6.

2. WMC and WMI

2.1. Weighted Model Counting

Definition 2 (Propositional formula)**.** Let $\mathbf{b} = \{ B_1, B_2, \dots, B_M \}$ be a set of M Boolean variables (or logical propositions), which can be combined in the usual way using logical connectives $\neg$, $\wedge$ and $\vee$, producing *formulas of propositional logic*. We call a *literal* either a Boolean variable or its negation, and denote with $\mathcal{L}_\mathbf{b} = \{ B_1, \neg B_1, B_2, \neg B_2, \dots, B_M, \neg B_M \}$ the set of all literals over the set $\mathbf{b}$.

Symbol $\mathbf{b}$ will be used throughout the paper to denote the set $\{ B_1, B_2, \dots, B_M \}$ of Boolean variables. Without loss of generality, we regard the sets of (Boolean, real, integer) variables as ordered sets (cf. Definitions 4 and 8).

The set of Boolean (truth) values will be denoted by $\mathbb{B} = \{ \bot, \top \}$. In order to assign a truth value to a formula, we introduce the concept of an *interpretation*.

Definition 3 (Interpretation of a propositional formula)**.** A *total interpretation* of the Boolean variables in $\mathbf{b}$ is any mapping from set $\mathbf{b}$ to the set $\mathbb{B}$. We require this mapping to commute with logical connectives in the usual way, so that it can be extended to any propositional formula built from variables in $\mathbf{b}$. A propositional formula ϕ is said to be *true* under the interpretation I if $I(\phi) = \top$ and *false* otherwise.

Closely connected to the notion of interpretation is that of a *model*.

Definition 4 (Model of a propositional formula)**.** Let I be an interpretation and ϕ a propositional formula over $\mathbf{b}$, such that $I(\phi) = \top$. We say that the M-tuple

$$\mathfrak{M}(I) = (I(B_1), I(B_2), \dots, I(B_M)) \in \mathbb{B}^M$$

is a *model* of ϕ associated with interpretation I. We denote the *set of all models* of the propositional formula ϕ by $\mathcal{M}(\phi) = \{ \mathfrak{M}(I) \mid I(\phi) = \top \}$.

In the WMC literature, the model of a propositional formula associated with an interpretation I is traditionally defined as a subset of $\mathcal{L}_\mathbf{b}$ containing literals that are true under this interpretation [1, 21]. Any such subset $\mathfrak{M}^\mathcal{L}(I) = \{ \ell_1, \ell_2, \dots, \ell_M \}$, where $\ell_i = \operatorname{ite}(I(B_i), B_i, \neg B_i)$ [1] for all $i = 1, 2, \dots, M$, uniquely defines the M-tuple $(I(B_1), I(B_2), \dots, I(B_M)) \in \mathbb{B}^M$ used in the previous Definition, and vice versa.[2]

Using this notation, the well known Boolean satisfiability problem (SAT) is expressed as the problem of determining whether $\mathcal{M}(\phi) = \emptyset$. Its counting counterpart (#SAT) is expressed as determining the exact number of elements in $\mathcal{M}(\phi)$.

Definition 5 (WMC)**.** Let $\mathbf{b}$ be a set of M Boolean variables, and ϕ be a propositional formula over $\mathbf{b}$. Furthermore, let $w^\mathcal{L} \colon \mathcal{L}_\mathbf{b} \to \mathbb{R}_{\geq 0}$ be a weight function of Boolean literals. Then the *weighted model count (WMC)* of the formula ϕ is given by:

$$\operatorname{WMC}\left(\phi, w^\mathcal{L} \mid \mathbf{b}\right) = \sum_{\mathfrak{M} \in \mathcal{M}(\phi)} \prod_{\ell \in \mathfrak{M}^\mathcal{L}} w^\mathcal{L}(\ell). \tag{1}$$

[1] The function symbol 'ite' denotes the *if-then-else* function: if the first argument is $\top$ (true) the second argument is returned, else the third argument is returned.

[2] Note that our definition of an interpretation slightly differs from the one given in [21]. In [21] an interpretation is an instantiation of variables (function image) that satisfies a logic formula. In contrast, we define an interpretation as a mapping (function) from variables to the set $\mathbb{B}$. The definitions of a model in [21] and this paper do coincide again.

For simplicity of exposition, we assume the weight function to be non-negative, which is also justified by weight functions used in practice. The importance of WMC for probabilistic inference cannot be overstated, and is thoroughly investigated in [1, 4]. Further interesting generalizations of WMC to semirings other than the $\mathbb{R}$-semiring are discussed in [21].

2.2. Weighted Model Integration

Many applications require probabilistic inference in continuous domains. In order to capture these applications, the task of weighted model counting has been extended to weighted model integration [5]. The first step is a definition of a logical theory which combines Boolean and continuous variables. To this end we follow the definition in [11] (a more formal definition can be found in [22]).

Definition 6 (SMT formula)**.** Let $\mathbf{b} = \{ B_1, B_2, \ldots, B_M \}$ be a set of M Boolean variables, and $\mathbf{x} = \{ X_1, X_2, \ldots, X_N \}$ be a set of N real variables. An *atomic formula* is either a Boolean variable (**logical proposition**) from set $\mathbf{b}$, or a well-formed arithmetical statement (**real arithmetical proposition**) consisting of variables from $\mathbf{x}$, real numbers and symbols $+$, $\cdot$, $\wedge$, and $\leq$, having standard interpretation as real addition, multiplication, exponentiation, and less-than inequality, respectively. Atomic formulas are combined using logical connectives $\neg$, $\land$ and $\lor$, producing so-called *SMT formulas*.

Analogously to the Boolean case, aymbol $\mathbf{x}$ will be used throughout the paper to denote the set $\{ X_1, X_2, \ldots, X_N \}$ of real (continuous) variables.

We note that any real arithmetical proposition θ can be written in the logically equivalent form $\theta'(X_1, X_2, \ldots, X_N) \leq 0$. We denote by $\hat{\theta}$ a function from $\mathbb{R}^N$ to $\mathbb{R}$ encoded by proposition θ'. Based on the restrictions posed on this function, we distinguish, among others, SMT($\mathcal{LRA}$) formulas ($\hat{\theta}$ is a linear function), SMT($\mathcal{NRA}$) formulas ($\hat{\theta}$ is a polynomial function) and SMT($\mathcal{RA}$) formulas ($\hat{\theta}$ is unrestricted).

Definition 7 (Interpretation of an SMT formula)**.** Let an SMT formula be built over the Boolean variables in $\mathbf{b}$ and continuous variables in $\mathbf{x}$. A *total interpretation* of the variables in $\mathbf{b}$ and $\mathbf{x}$ is a pair $I = (I_{\mathbf{b}}, I_{\mathbf{x}})$, where $I_{\mathbf{b}}$ is a mapping from $\mathbf{b}$ to $\mathbb{B}$ and $I_{\mathbf{x}}$ is a mapping from $\mathbf{x}$ to $\mathbb{R}$.

The logical value of an atomic formula θ under the interpretation I, is defined as $I(\theta) := I_{\mathbf{b}}(\theta)$ if θ is a logical proposition. In case of θ being a real arithmetical proposition, we define $I(\theta) := \top$ if the inequality $\hat{\theta}(I_{\mathbf{x}}(X_1), \ldots, I_{\mathbf{x}}(X_N)) \leq 0$ holds, and $I(\theta) := \bot$ otherwise. Requiring an interpretation to commute with logical connectives in the usual way extends the definition of interpretation to any SMT formula.

The mappings $I_{\mathbf{b}}$ and $I_{\mathbf{x}}$ from the previous definition are called *partial interpretations* of Boolean and continuous variables, respectively. Analogously to the purely Boolean case, we define models of SMT formulas as $(M + N)$-tuples.

Definition 8 (Model of an SMT formula)**.** Let $I = (I_{\mathbf{b}}, I_{\mathbf{x}})$ be an interpretation and ϕ an SMT formula over variables in $\mathbf{b}$ and $\mathbf{x}$ such that $I(\phi) = \top$. We say that

$$\begin{aligned} \mathfrak{M}(I) &= (\mathfrak{M}_{\mathbf{b}}(I), \mathfrak{M}_{\mathbf{x}}(I)) \\ &= \left((I_{\mathbf{b}}(B_i))_{i=1}^{M}, \left(I_{\mathbf{x}}(X_j) \right)_{j=1}^{N} \right) \in \mathbb{B}^M \times \mathbb{R}^N \end{aligned}$$

is a *model* of formula ϕ associated to interpretation I. The *set of all models* is denoted again by $\mathcal{M}(\phi)$.

The projection of $\mathcal{M}(\phi)$ to $\mathbb{B}^M$ is denoted by

$$\mathcal{M}_{\mathbf{b}}(\phi) = \left\{ b \in \mathbb{B}^M \mid \text{there is } x \in \mathbb{R}^N \text{ such that } (b, x) \in \mathcal{M}(\phi) \right\},$$

and analogously by $\mathcal{M}_{\mathbf{x}}(\phi)$ its projection to $\mathbb{R}^N$. These sets contain *partial models* of a formula and are used below in the definition of weighted model integration. Furthermore, for any $b \in \mathbb{B}^M$, set denoted by

$$\mathcal{M}_{\mathbf{x}}(\phi)/b = \left\{ x \in \mathbb{R}^N \mid (b, x) \in \mathcal{M}(\phi) \right\}$$

consists of elements $x \in \mathcal{M}_{\mathbf{x}}(\phi)$ which extend partial model $b \in \mathcal{M}_{\mathbf{b}}(\phi)$ to a total model $(b, x) \in \mathcal{M}(\phi)$.

Definition 9 (WMI)**.** Let **b** be a set of M Boolean variables, **x** a set of N real variables, and ϕ an SMT formula over **b** and **x**. Let $w \colon \mathbb{B}^M \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ be a weight function of Boolean and real variables. For any $b \in \mathbb{B}$, a function $w_b \colon \mathbb{R}^N \to \mathbb{R}$ is defined with $w_b(x) = w(b, x)$, for all $x \in \mathbb{R}^N$. Assume that for all $b \in \mathcal{M}_{\mathbf{b}}(\phi)$, the functions w_b are Riemann integrable on the sets $\mathcal{M}_{\mathbf{x}}(\phi)/b$, respectively. We define the *weighted model integral (WMI)* of a formula ϕ with regards to the weight function w by:

$$\mathrm{WMI}\,(\phi, w \mid \mathbf{b}, \mathbf{x}) = \sum_{b \in \mathcal{M}_{\mathbf{b}}(\phi)} \int_{x \in \mathcal{M}_{\mathbf{x}}(\phi)/b} w_b(x)\, dx_1\, dx_2 \cdots dx_N. \tag{2}$$

3. Measure theoretic WMC and WMI

As a central contribution, we introduce variants of both WMC (Definition 5) and WMI (Definition 9) based on measure theory, introducing **measures of weighted propositional logic and SMT formulas**. We proceed to prove that they generalize classical WMC and WMI based on summation and Riemann integration. This measure theoretic formulation of WMC and WMI yields an elegant proof of congruence of these two concepts in the special case of a purely Boolean domain.

A formulation that treats Boolean and real variables on equal footing and leads to the congruence of WMC and WMI has also been presented in [16], where the authors reduce weighted model integration to model integration [23]. However, the reduction is performed by transforming the summation over Boolean variables to a Riemann integration over real variables without relying on the more powerful and expressive Lebesgue integration.

Prior to formulating weighted model counting and integration as a measure theoretic problem, we give a brief introduction to measure theory. A complete formal excursion on the measure theoretic concepts essential to this paper is provided in Appendix A.

3.1. An Appetizer of Measure Theory

Let us assume we have two real numbers a and b. We would like to know how far these two numbers are apart. In other words, we would like to know the length l of the segment S delimited by a and b. In Euclidean geometry, the length l is simply given by $l = |b - a|$. Now, instead of viewing l as the length of the segment S, we can also regard l as the size of the set of points that make up S.

Measure theory generalizes the concepts of length, area and volume by answering the question *'how big is a specific set?'* This is done by systematically assigning a positive real number

Table 1: Glossary of technical terms used in measure theory.

σ-algebra (Def. A27)	Lebesgue measure (Def. A31)
measurable space (Def. A27)	measurable function (Def A32)
Borel σ-algebra (Def. A28)	simple function (Def. A34)
countably additive (Def. A28)	μ-almost everywhere (Def. A39)
measure (Def. A29)	product measure (Theo. A41)
counting measure (Def. A30)	probability space (Def. A43)

to a given set. A set is called measurable if such a number can actually be assigned. In Euclidean geometry, a measure of particular importance is the Lebesgue measure, which assigns the conventional Euclidean length, volume, and hypervolume to measurable subsets of the N-dimensional Euclidean space $\mathbb{R}^N$.

Furthermore, measure theory does also provide the axiomatic formulation of probability theory, as developed by Kolmogorov [24]. More precisely, axiomatic probability theory is formulated in terms of measures that assign to the whole set (domain of definition) size 1, and where events are interpreted as measurable subsets, whose probability is given by the same measure. Also, in this setting the expectation of a random variable corresponds to the Lebesgue integral of a function computed over the probability measure.

On the other hand, in the context of model counting (#SAT), we want to determine/measure the size of the set of satisfying assignments to a propositional logic formula.

For the uninitiated reader we provide in Table 1 a glossary of technical terms used in measure theory and give the relevant pointers to their introduction in Appendix A.

3.2. Measure Theoretic WMC

In order to embed WMC into measure theory (using Lebesgue integration), a slight adjustment to Definition 5 is in order. It is more convenient to define a weight function over the set $\mathbb{B}^M$, similarly to Definition 9, instead of over the set of literals $\mathcal{L}_{\mathbf{b}}$. To this end, we transform the given weight function $w^{\mathcal{L}}: \mathcal{L}_{\mathbf{b}} \to \mathbb{R}_{\geq 0}$ over literals to an equivalent weight function $w: \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ over $\mathbb{B}^M$, as follows: for any $b = (b_1, b_2, \dots, b_M) \in \mathbb{B}^M$, let

$$w(b) = \prod_{i=1}^{M} \text{ite}\left(b_i, w^{\mathcal{L}}(B_i), w^{\mathcal{L}}(\neg B_i)\right). \tag{3}$$

Equation (1) now becomes:

$$\text{WMC}\left(\phi, w^{\mathcal{L}} \mid \mathbf{b}\right) = \sum_{\mathfrak{M} \in \mathcal{M}(\phi)} w(\mathfrak{M}).$$

Notice that this expression already looks 'Lebesguean'. Indeed, we only need to specify the components of an appropriate measure space.

Proposition 10. $\left(\mathbb{B}^M, \mathcal{P}(\mathbb{B}^M), \mu\right)$ *is a measure space, where* $\mathcal{P}(\mathbb{B}^M)$ *is a power set of* $\mathbb{B}^M$ *and* $\mu: \mathcal{P}(\mathbb{B}^M) \to [0, +\infty]$ *is a counting measure.*

Proof. Any set together with a counting measure on its power set defines a measure space. □

We are now in the position to express the weighted model count in measure theoretic terms.

Definition 11 (Lebesgue WMC)**.** Let $\mathbf{b}$ be a set of M Boolean variables, and ϕ a propositional formula over $\mathbf{b}$. Furthermore, let $w\colon \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ be a weight function. The *Lebesgue weighted model count (*L−WMC*)* of the formula ϕ with respect to the weight w is defined by:

$$\text{L-WMC}\,(\phi, w) = \int_{\mathcal{M}(\phi)} w\, d\mu\,.$$

The integral in the previous definition is well-defined, because w is obviously bounded (as the set $\mathbb{B}^M$ is finite) and it is trivially measurable (as the whole power set of $\mathbb{B}^M$ is a σ-algebra).

Theorem 12. *Let* $\mathbf{b}$ *be a set of M Boolean variables, and* ϕ *be a propositional formula over* $\mathbf{b}$*. Furthermore, let* $w^{\mathcal{L}}\colon \mathcal{L}_{\mathbf{b}} \to \mathbb{R}_{\geq 0}$ *be a* weight function *of Boolean literals and* $w\colon \mathbb{B}^M \to \mathbb{R}$ *be constructed from* $w^{\mathcal{L}}$ *as in Equation* (3)*. Then:*

$$\text{L-WMC}\,(\phi, w) = \text{WMC}\left(\phi, w^{\mathcal{L}} \mid \mathbf{b}\right).$$

Proof. Since $\mathbb{B}^M$ is a finite set, w is a simple function. $[\![\cdot]\!]$ will denote the Iverson bracket, which evaluates to 1 if its argument is satisfied, and 0 otherwise [25, 26].

$$\begin{aligned}
\text{L-WMC}\,(\phi, w) = \int_{\mathcal{M}(\phi)} w\, d\mu &= \int_{\mathbb{B}^M} \left(w \cdot \mathbb{1}_{\mathcal{M}(\phi)}\right) d\mu \\
&= \int_{\mathbb{B}^M} \left(\left(\sum_{b \in \mathbb{B}^M} w(b) \cdot \mathbb{1}_{\{\, b\,\}}\right) \cdot \mathbb{1}_{\mathcal{M}(\phi)}\right) d\mu \\
&= \int_{\mathbb{B}^M} \left(\sum_{b \in \mathbb{B}^M} w(b) \cdot \mathbb{1}_{\mathcal{M}(\phi) \cap \{\, b\,\}}\right) d\mu \\
&= \sum_{b \in \mathbb{B}^M} w(b) \cdot \mu(\mathcal{M}(\phi) \cap \{\, b\,\}) \\
&= \sum_{b \in \mathbb{B}^M} w(b) \cdot [\![b \in \mathcal{M}(\phi)]\!] \\
&= \sum_{b \in \mathcal{M}(\phi)} w(b) = \text{WMC}\left(\phi, w^{\mathcal{L}} \mid \mathbf{b}\right). \qquad \square
\end{aligned}$$

This proves that the newly defined Lebesgue weighted model count, based on measure theory, coincides with the classical weighted model count from Definition 5. This result is the first step towards a measure theoretic formulation of WMI.

3.3. Measure Theoretic WMI

We now turn to introducing an appropriate measure space for the hybrid domain consisting of Boolean and real variables, and proving the central result of this paper. In the following, $\mathcal{B}(\mathbb{R}^N)$ denotes the Borel σ-algebra on $\mathbb{R}^N$ from Definition A28 and λ^N the Lebesgue measure on $\mathbb{R}^N$ from Definition A31. The exponent in λ^N shall be omitted for simplicity, when the dimension of the real space is clear from context.

Proposition 13. $\left(\mathbb{B}^M \times \mathbb{R}^N, \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N), \mu \times \lambda\right)$ *is a measure space, which is a product of the measure space* $\left(\mathbb{B}^M, \mathcal{P}(\mathbb{B}^M), \mu\right)$ *from Proposition 10 with measure space* $\left(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \lambda\right)$*.*

Proof. See Theorem A41. □

Next we introduce the technical concept of *measurability* of an SMT formula. Say that an SMT formula ϕ is *measurable* if its set of models $\mathcal{M}(\phi)$ is a mesurable set in the measure space from proposition 13.

Lemma 14. *SMT($\mathcal{LRA}$), SMT($\mathcal{NRA}$) and SMT($\mathcal{RA}$) formulas are measurable.*

Proof. Linear functions, polynomials and generally all real functions obtained by means of addition, multiplication and exponentiation of real variables and constants, are continuous. Hence, they are Borel measurable (see Example A33). Now note that the set of models for any real arithmetical proposition θ is $\hat{\theta}^{-1}([-\infty, 0])$. Therefore, these sets are Lebesgue measurable (cf. Definition A31). The set of models of logical proposition is always measurable, since σ-algebra on $\mathbb{B}^M$ is the whole power set $\mathcal{P}(\mathbb{B}^M)$.

The set of models of any formula from the above theories is now obtained as a (possibly complement of) finite union and intersection of products of models for the Boolean and real parts of the formula. By definition they remain elements of the product σ-algebra, i.e. they are measurable. □

We have set the stage for the definition of the measure theoretic weighted model integral.

Definition 15 (Lebesgue WMI)**.** Let $\mathbf{b}$ be a set of M Boolean variables, $\mathbf{x}$ a set of N real variables and ϕ an SMT formula over $\mathbf{b}$ and $\mathbf{x}$. Furthermore, let $w\colon \mathbb{B}^M \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ be a weight function of Boolean and real variables. Assume that the formula ϕ is measurable and the function w is integrable with regards to the product measure $\mu \times \lambda$ on $\mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N)$ from Proposition 13. We define the *Lebesgue weighted model integral (*L−WMI*)* of the formula ϕ with respect to the weight w as:

$$\text{L}-\text{WMI}\,(\phi, w) = \int_{\mathcal{M}(\phi)} w\, d(\mu \times \lambda).$$

For weight functions that are not Riemann integrable but Lebesgue, Definition 15 provides an alternative to Defintion 9 for the weighted model intergral. On the other hand, in case of Riemann integrable weight functions, WMI and L−WMI are equal.

Theorem 16. *Under the assumptions of Definition 9, the following equality holds:*

$$\text{L}-\text{WMI}\,(\phi, w) = \text{WMI}\,(\phi, w \mid \mathbf{b}, \mathbf{x})\ .$$

Proof. For each $b \in \mathbb{B}^M$, functions w_b are by assumption Riemann integrable over sets $\mathcal{M}_{\mathbf{x}}(\phi)/b$, respectively. This implies that the sets $\mathcal{M}_{\mathbf{x}}(\phi)/b$ are Borel measurable (they are bounded sections), and that the functions w_b are Lebesgue integrable over these sets, respectively (see Theorem A38). Furthermore, the set $\mathbb{B}^M$ is finite, and the following identities clearly hold:

$$\mathcal{M}(\phi) = \bigcup\nolimits_{\beta \in \mathbb{B}^M} \{\beta\} \times \mathcal{M}_{\mathbf{x}}(\phi)/\beta,$$
$$w(b, x) = \sum\nolimits_{\beta \in \mathbb{B}^M} [\![\beta = b]\!] \cdot w_b(x).$$

We conclude that the formula ϕ is measurable (as its model is a union of Borel measurable sets, hence Borel measurable itself), and that the function w is integrable with regards to the product measure $\mu \times \lambda$ (as a linear combination of integrable functions, see Proposition A37). Finally, we obtain:

$$\begin{aligned}
\text{L-WMI}(\phi, w) = \int_{\mathcal{M}(\phi)} w \, d(\mu \times \lambda) &= \int_{\mathbb{B}^M \times \mathbb{R}^N} \left(w \cdot \mathbb{1}_{\mathcal{M}(\phi)} \right) d(\mu \times \lambda) \\
&= \int_{\mathbb{B}^M} \left(\int_{\mathbb{R}^N} \left(w \cdot \mathbb{1}_{\mathcal{M}(\phi)} \right)_b d\lambda \right) d\mu \qquad \text{(Theorem A42)} \\
&= \int_{\mathbb{B}^M} \left(\int_{\mathbb{R}^N} w_b \cdot [\![b \in \mathcal{M}_{\mathbf{b}}(\phi)]\!] \cdot \mathbb{1}_{\mathcal{M}_{\mathbf{x}}(\phi)/b} \, d\lambda \right) d\mu \\
&= \int_{\mathbb{B}^M} \left(\int_{\mathbb{R}^N} w_b \cdot \mathbb{1}_{\mathcal{M}_{\mathbf{x}}(\phi)/b} \, d\lambda \right) \cdot \mathbb{1}_{\mathcal{M}_{\mathbf{b}}(\phi)} \, d\mu \\
&= \sum_{b \in \mathcal{M}_{\mathbf{b}}(\phi)} \int_{\mathcal{M}_{\mathbf{x}}(\phi)/b} w_b \, d\lambda \\
&= \text{WMI}(\phi, w \mid \mathbf{b}, \mathbf{x}). \qquad \square
\end{aligned}$$

Both, Theorem 12 and Theorem 16, state that the weighted model count/integral of a formula is equal to the Lebesgue integral of the weight function over the set of models of a formula. This unification enables us to elegantly prove that WMC is a special case of WMI.

Corollary 17. *Let* **b** *be a set of M Boolean variables, and ϕ be a propositional formula over* **b**. *Furthermore, let $w^{\mathcal{L}} \colon \mathcal{L}_{\mathbf{b}} \to \mathbb{R}_{\geq 0}$ be a weight function of Boolean literals and $w \colon \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ be constructed from $w^{\mathcal{L}}$ as in Equation* (3). *Then:*

$$\text{WMC}\left(\phi, w^{\mathcal{L}} \mid \mathbf{b}\right) = \text{WMI}(\phi, w \mid \mathbf{b}, \emptyset) \tag{4}$$

Proof. From the point of view of weighted model integration, this presents a degenerate case with no real variables ($\mathbf{x} = \emptyset$). The space reduces to the Boolean space $\mathbb{B}^M$ and the measure reduces to the Boolean measure μ, i.e. $\mathbb{B}^M \times \mathbb{R}^0 = \mathbb{B}^M$ and $\mu \times \lambda^0 = \mu$. Plugging this into Theorem 16, together with Theorem 12, yields

$$\text{WMI}(\phi, w \mid \mathbf{b}, \emptyset) = \int_{\mathcal{M}(\phi)} w \, d\mu = \text{WMC}(\phi, w^{\mathcal{L}} \mid \mathbf{b}). \qquad \square$$

L−WMI can now easily be extended to domains including integer variables, besides Boolean and real ones. The construction is completely analogous to the one presented in this section. The appropriate measure space is

$$\left(\mathbb{B}^M \times \mathbb{R}^N \times \mathbb{Z}^K,\ \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N) \times \mathcal{P}(\mathbb{Z}^K),\ \mu \times \lambda \times \xi\right), \tag{5}$$

where ξ is the counting measure on $\mathcal{P}(\mathbb{Z}^K)$, and analogous results as in Theorem 16 and Corollary 17 hold.

4. Weight Functions as Measures

A major application of WMC and WMI is to be found inside probabilistic inference tasks. There a weight function can be regarded as a probability density function (PDF). This enables us to define a measure (i.e. a probability) on the underlying space directly from a weight function. In this section we consider the weighted model count/integral of a logical formula in this probabilistic setting[3]. Under these assumptions, WMC and WMI equal simply the probability of the set of models. The integration process gets encapsulated into the construction of the probability (cf. celebrated Radon-Nikodym derivative [32, Theorem 6.2.3]). This approach can be extended beyond probabilistic measures, that is, to any finite measure which represents a weight function. It is also suitable for hybrid domains with integers, in the manner explained at the end of the previous section.

4.1. Weighted Model Counting as Measure

Let again $\mathbf{b} = \{B_1, B_2, \ldots, B_M\}$ be a set of M Boolean variables which form the basis of propositional logic. Assume that weight function $w\colon \mathbb{B}^M \to [0,1]$ is a PDF on $\mathbb{B}^M$, i.e. $\sum_{b\in\mathbb{B}^M} w(b) = 1$ holds. The next proposition introduces a natural probability which arises from the weight function w. We refer to it as a *probability associated to the weight function w*. As before, μ denotes the counting measure on $\mathcal{P}(\mathbb{B}^M)$.

Proposition 18. *Let $w\colon \mathbb{B}^M \to [0,1]$ be a weight function such that $\sum_{b\in\mathbb{B}^M} w(b) = 1$. For any $B \subset \mathbb{B}^M$, let $\eta\colon \mathcal{P}(\mathbb{B}^M) \to [0,1]$ be given with*

$$\eta(B) = \sum\nolimits_{b\in B} w(b) = \int_B w \, d\mu.$$

Then $\left(\mathbb{B}^M, \mathcal{P}(\mathbb{B}^M), \eta\right)$ is a probability space.

Proof. Follows trivially from the definition of η and the properties of w. □

As in Section 2.1, we describe how a probabilistic weight function of Boolean literals can naturally be transformed into a PDF on $\mathbb{B}^M$. Let $w^{\mathcal{L}}\colon \mathcal{L}_{\mathbf{b}} \to [0,1]$ be a function that, for every $i = 1, 2, \ldots, M$, satisfies

$$w^{\mathcal{L}}(B_i) + w^{\mathcal{L}}(\neg B_i) = 1.$$

Using the same construction as in Equation (3), we get a function $w\colon \mathbb{B}^M \to [0,1]$ satisfying

$$\sum_{b\in\mathbb{B}^M} w(b) = \sum_{b\in\mathbb{B}^M} \prod_{i=1}^{M} \mathrm{ite}\left(b_i, w^{\mathcal{L}}(B_i), w^{\mathcal{L}}(\neg B_i)\right) = \prod_{i=1}^{M} \left(w^{\mathcal{L}}(B_i) + w^{\mathcal{L}}(\neg B_i)\right) = 1. \tag{6}$$

Proposition 18 now produces a probability η associated with $w^{\mathcal{L}}$. Factorization over literals indicates their independence with regards to the probability η.

The weighted model count is now obtained by simply measuring the size of the set of models, using the just introduced probability.

Theorem 19. *Let* $\mathbf{b}$ *be a set of M Boolean variables, and ϕ be a propositional formula over* $\mathbf{b}$. *Let $w^{\mathcal{L}}\colon \mathcal{L}_{\mathbf{b}} \to [0,1]$ be a probabilistic weight function of Boolean literals and $w\colon \mathbb{B}^M \to [0,1]$ be a PDF constructed from $w^{\mathcal{L}}$ as in Equation* (3). *Furthermore, let η be the probability associated to the weight function w. Then:*

$$\mathrm{L{-}WMC}\left(\phi, w^{\mathcal{L}} \mid \mathbf{b}\right) = \eta\left(\mathcal{M}(\phi)\right). \tag{7}$$

Proof. Follows directly from Definition 11 and Proposition 18. □

[3]The discussion of WMI, in this paper, is limited to finitely many variables. In probabilistic logic programming this is also called the *finite support condition* [27], which is exposed in more more detail in [28] (for the Boolean case) and in [29] (for the discrete-continuous case). A possible avenue for future research is an extension to infinitely (including uncountably) many variables in the special case of WMI with probability measures, cf. [27], [30, Theorem 6.18], and [31].

4.2. Weighted Model Integration as Measure

We extend the discussion from the last section to the hybrid domain. Let again $\mathbf{x} = \{X_1, X_2, \ldots, X_N\}$ be the set of real variables. Given a PDF $w\colon \mathbb{B}^M \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$, we obtain the probability $\nu\colon \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N) \to [0, 1]$ defined with

$$\nu(E) = \int_E w \, d(\mu \times \lambda), \tag{8}$$

for every $E \in \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N)$. Because of the form of this measure, a hybrid-domain analogue of Theorem 19 is obtained effortlessly.

Theorem 20. *Let* $\mathbf{b}$ *be a set of M Boolean variables,* $\mathbf{x}$ *a set of N real variables and* ϕ *a measurable SMT formula over* $\mathbf{b}$ *and* $\mathbf{x}$*. Furthermore, let* $w\colon \mathbb{B}^M \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ *be a weight function of Boolean and real variables. If w is a PDF on* $\mathbb{B}^M \times \mathbb{R}^N$ *defining probability* ν *given by* (8)*, then:*

$$\text{L-WMI}\,(\phi, w) = \nu\,(\mathcal{M}(\phi))\,.$$

Proof. Follows directly from Definition 15. □

In the following we are concerned with the factorization of a weight function into separate parts over Boolean and continuous spaces, respectively. This discussion is of interest, as probabilities on Boolean and continuous spaces can be combined together using the product measure construction. We begin with general setting, and later comment on an important special case, where the weight function fully factorizes.

Any weight function $w\colon \mathbb{B}^M \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ can be partially factorized such that the equality

$$w(b, x) = w_{\mathbf{b}}(b) \cdot w_{\mathbf{x}}^{b}(x) \tag{9}$$

holds for all $b \in \mathbb{B}^M$ and $x \in \mathbb{R}^N$ (note the dependency of the second factor on b). The function $w_{\mathbf{b}}\colon \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ is the Boolean part of the function w and, for each $b \in \mathbb{B}^M$, the function $w_{\mathbf{x}}^{b}\colon \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is a piece of the continuous part. This factorization is generally not unique. For instance, every non-zero $c \in \mathbb{R}$ defines a simple factorization, given with $w_{\mathbf{b}}(b) = c$ and $w_{\mathbf{x}}^{b}(x) = \frac{1}{c} w(b, x)$, for all $b \in \mathbb{B}^M$ and $x \in \mathbb{R}^N$. However, in the probabilistic setting, partial factorization is essentially unique:

Lemma 21. *Let w be a PDF on* $\mathbb{B}^M \times \mathbb{R}^N$*. Then there are* ***unique PDFs*** $w_{\mathbf{b}}$ *on* $\mathbb{B}^M$ *and* $w_{\mathbf{x}}^{b}$ *on* $\mathbb{R}^N$*, for each* $b \in \mathbb{B}^M$*, such that Equality* (9) *holds for every* $b \in \mathbb{B}^M$ *and for* λ*-almost every* $x \in \mathbb{R}^N$ *(cf. Definition A39).*

Proof. For each $b \in \mathbb{B}^M$, denote again with $w_b\colon \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ a function given with $w_b(x) = w(b, x)$. Define the function $w_{\mathbf{b}}\colon \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ with $w_{\mathbf{b}}(b) = \int_{\mathbb{R}^N} w_b \, d\lambda$, for every $b \in \mathbb{B}^M$. Now for each $b \in \mathbb{B}^M$, define the functions $w_{\mathbf{x}}^{b}\colon \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ with $w_{\mathbf{x}}^{b}(x) = \frac{w(b,x)}{w_{\mathbf{b}}(b)}$ if $w_{\mathbf{b}}(b) \neq 0$, and $w_{\mathbf{x}}^{b}(x) = 0$ otherwise, for every $x \in \mathbb{R}^N$. In the former case, Equation (9) obviously holds. In the latter case, from Lemma A40 we conclude that $w_b(x) = w(b, x) = 0$ for λ-almost every $x \in \mathbb{R}^N$, and therefore Equation (9) holds λ-almost everywhere on $\mathbb{R}^N$.

Using the fact that w is a PDF together with Theorem A42, we prove that $w_{\mathbf{b}}$ is a PDF as well:

$$\int_{\mathbb{B}^M} w_{\mathbf{b}} \, d\mu = \int_{\mathbb{B}^M} \left(\int_{\mathbb{R}^N} w(b, x) \, d\lambda \right) d\mu = \int_{\mathbb{B}^M \times \mathbb{R}^N} w \, d\,(\mu \times \lambda) = 1. \tag{10}$$

Proving that $w_{\mathbf{x}}^b$ is a PDF, for each $b \in \mathbb{B}^M$, is trivial. From

$$1 = \int_{\mathbb{R}^N} w_{\mathbf{x}}^b \, d\lambda = \int_{\mathbb{R}^N} \frac{w(b,x)}{w_{\mathbf{b}}(b)} \, d\lambda = \frac{1}{w_{\mathbf{b}}(b)} \int_{\mathbb{R}^N} w_b \, d\lambda$$

it follows that $w_{\mathbf{b}}$ is unique, and then $w_{\mathbf{x}}^b$ as well. □

As a consequence, we can split the probability ν from Equation (8) into Boolean and continuous parts. For a given PDF w on $\mathbb{B}^M \times \mathbb{R}^N$, we first find its unique factors from Lemma 21, that is the PDFs $w_{\mathbf{b}}$ on $\mathbb{B}^M$ and $w_{\mathbf{x}}^b$ on $\mathbb{R}^N$, for each $b \in \mathbb{B}^M$. Each function $w_{\mathbf{x}}^b$ defines a probability τ^b on $\mathcal{B}(\mathbb{R}^N)$ given with

$$\tau^b(E) = \int_E w_{\mathbf{x}}^b \, d\lambda, \tag{11}$$

for every set $E \in \mathcal{B}(\mathbb{R}^N)$. Using a construction similar to that of the product measure in Theorem A41, the probability η associated to the weight function $w_{\mathbf{b}}$ (from Proposition 18) can be joined with the probabilities τ^b, $b \in \mathbb{B}^M$, in order to obtain a single probability on $\mathbb{B}^M \times \mathbb{R}^N$.

Proposition 22. *Let η be a probability on $\mathcal{P}(\mathbb{B}^M)$. For every $b \in \mathbb{B}^M$, let τ^b be a probability on $\mathcal{B}(\mathbb{R}^N)$. Furthermore, let $\eta \times \tau \colon \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N) \to [0,1]$ denote a function given with*

$$(\eta \times \tau)(E) = \int_{\mathbb{B}^M} \tau^b(E_b) \, d\eta$$

for any set $E \in \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N)$, with notation the E_b being explained in Appendix A. The tuple

$$\left(\mathbb{B}^M \times \mathbb{R}^N, \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N), \eta \times \tau\right)$$

is a probability space.

Proof. The function $\eta \times \tau$ is countably additive, analogous to the proof of Theorem A41. The equality

$$(\eta \times \tau)\left(\mathbb{B}^M \times \mathbb{R}^N\right) = 1$$

follows immediately, since η and τ^b, for $b \in \mathbb{B}^M$, are all probabilities. □

The measure $\eta \times \tau$ is not a product measure, since τ alone has no meaning, yet. Below we describe an aforementioned important case when a weight function is fully factorized. This measure is indeed a product measure, offering a motivation for this notation. But first, let us rephrase the statement of Theorem 20, in accordance with our present discussion.

Corollary 23. *Let* $\mathbf{b}$ *be a set of M Boolean variables,* $\mathbf{x}$ *a set of N real variables, and ϕ a measurable SMT formula over variables in* $\mathbf{b}$ *and* $\mathbf{x}$*. Let $w \colon \mathbb{B}^M \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ be a weight function of Boolean and real variables, which is a PDF on $\mathbb{B}^M \times \mathbb{R}^N$. Now let $w_{\mathbf{b}} \colon \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ and, for each $b \in \mathbb{B}^M$, $w_{\mathbf{x}}^b \colon \mathbb{B}^M \to \mathbb{R}_{\geq 0}$ be unique PDFs such that $w(b,x) = w_{\mathbf{b}}(b) \cdot w_{\mathbf{x}}^b(x)$ holds for every $b \in \mathbb{B}^M$ and λ-almost every $x \in \mathbb{R}^M$. Furthermore, let η be a probability on $\mathcal{P}(\mathbb{B}^M)$ associated to the PDF $w_{\mathbf{b}}$ and, for each $b \in \mathbb{B}^M$, let τ^b be a probability on $\mathcal{B}(\mathbb{R}^N)$ associated to the PDF $w_{\mathbf{x}}^b$. Lastly, let $\eta \times \tau$ be a probability measure on $\mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N)$ associated to the PDF w obtained from the probabilities η and τ^b, for $b \in \mathbb{B}^M$, using Proposition 22. Then:*

$$\text{L-WMI}(\phi, w) = (\eta \times \tau)(\mathcal{M}(\phi)).$$

Proof. We prove that $(\eta \times \tau)$ equals the probability ν from Equation (8). Then the claim follows by Theorem 20. We note that $\eta(\{ b \}) = w_{\mathbf{b}}(b) \cdot \mu(\{ b \})$. Because of the factorization of the weight function w, for every $b \in \mathbb{B}^M$, we have $w_b = w_{\mathbf{b}}(b) \cdot w_{\mathbf{x}}^b$. Now for any $E \in \mathcal{P}(\mathbb{B}^M) \times \mathcal{B}(\mathbb{R}^N)$:

$$\begin{aligned}\nu(E) = \int_E w \, d\,(\mu \times \lambda) &= \int_{\mathbb{B}^M} \left(\int_{\mathbb{R}^N} w_b \cdot \mathbb{1}_{E_b} \, d\lambda \right) d\mu \\ &= \sum_{b \in \mathbb{B}^M} \left(\int_{E_b} w_{\mathbf{x}}^b \, d\lambda \right) \cdot w_{\mathbf{b}}(b) \cdot \mu(\{ b \}) \\ &= \int_{\mathbb{B}^M} \tau^b(E_b) \, d\eta = (\eta \times \tau)\,(E)\,. \end{aligned}$$

□

Lastly, we discuss the announced case of the full factorization of the weight function w. In practice, there is commonly a single PDF $w_{\mathbf{x}}$ associated to any $b \in \mathbb{B}^M$, i.e. equality

$$w(b, x) = w_{\mathbf{b}}(b) \cdot w_{\mathbf{x}}(x)$$

holds for every $b \in \mathbb{B}^M$ and λ-almost every $x \in \mathbb{R}^N$. Consequently, there is one probability measure τ on $\mathcal{B}(\mathbb{R}^N)$. Uniqueness of the product measure from Theorem A41 then implies that the probability space from Proposition 22 is actually a product of probability spaces $\left(\mathbb{B}^M, \mathcal{P}(\mathbb{B}^M), \eta \right)$ and $\left(\mathbb{R}^M, \mathcal{B}(\mathbb{R}^M), \tau \right)$. Result analogous to that of Corollary 23 is valid in this case.

5. The Usefulness of Measure Theoretic WMI for Computational Complexity Discussions

Early research on WMI was mainly concerned with encoding discrete-continuous problems and developing efficient solvers. Recently, however, more detailed studies of the computational complexity of WMI problems have emerged [16, 17, 20]. The main objective of the reported complexity studies is the delineation of tractable WMI problem classes: which WMI fragments are solvable in polynomial time (in contrast to the #P-hardness of general WMI problems)?

The presented computational complexity studies on WMI start with the observation that a WMI problem over Boolean and real variables can be reduced to an equivalent WMI problem over real variables only. Complexity results are then proven for these reduced WMI problems and consequently also hold for WMI problems with Boolean variables.

Reducing WMI problems to purely real-valued problems enabled Zeng and Van den Broeck [16] to present their complexity results for problems over different domains in a unified fashion. However, their method is inherently limited to Boolean- and real-valued WMI problems and does not extend to WMI problems with integers.

We are first going to sketch the reduction presented by Zeng and Van den Broeck [16] and point out its limitation to Bools and reals. Secondly, we are going to show how our measure theoretic formulation provides a more principled unification of discrete and continuous domains. This enables us to extend the discussion in Zeng and Van den Broeck [16] to cover, in addition to Booleans and reals, integer variables as well.

5.1. Reduction of WMI problems to purely real-valued problems

The reduction of WMI problems over Booleans and reals to purely real-valued problems was formally presented in Propostion 3.4 in [16]:

Proposition 24. *For each problem* $\mathrm{WMI}(\phi, w \mid \mathbf{x}, \mathbf{b})$ *there exists an equivalent problem* $\mathrm{WMI}(\phi', w' \mid \mathbf{x}')$ *without Boolean variables* $\mathbf{b}$ *such that*

$$\mathrm{WMI}(\phi, w \mid \mathbf{x}, \mathbf{b}) = \mathrm{WMI}(\phi', w' \mid \mathbf{x}')$$

and the primal graphs of ϕ *and* ϕ' *are isomorphic.*

The proposition above uses the following definition of primal graph of SMT formulas ([16, Definition 3.2]).

Definition 25 (Primal graph of an SMT formula)**.** The primal graph of an SMT($\mathcal{LRA}$) formula (in conjunctive normal form) is an undirected graph whose vertices are all variables and whose edges connect any two variables that appear in the same clause.

The structure of the primal graph is the key property in determining the complexity class of a WMI problem [16]. More concretely, a WMI problem on formula ϕ falls into the same computational complexity class as an equivalent WMI problem on formula ϕ', if ϕ and ϕ' are isomorphic.

In [16], the authors prove Proposition 24 by replacing Boolean variables with real valued variables. More precisely, in an SMT formula ϕ they replace a Boolean variable b using fresh atomic SMT formulas $(0{<}x_b \wedge x_b{<}1)$ and $(-1{<}x_b \wedge x_b{<}0)$ for the negation. The Boolean-free SMT formula is called ϕ'. Furthermore, they introduce a new weight function that depends on x_b instead of b:

$$w'(\mathbf{x}, x_b) = \begin{cases} w'_{\neg b}(\mathbf{x}, x_b) = w(\mathbf{x}, \neg b), \text{ if } -1 < x_b < 0; \\ w'_b(\mathbf{x}, x_b) = w(\mathbf{x}, b), \text{ if } 0 < x_b < 1; \\ 0, \text{else.} \end{cases} \tag{12}$$

This allows for rewriting the weighted model integral as a pure Riemann integration without summing out Boolean variables:

$$\mathrm{WMI}(\phi, w \mid \mathbf{x}, \mathbf{b}) = \int_{\phi'(\mathbf{x}, x_b)} w'(\mathbf{x}, x_b)\, d\mathbf{x}\, dx_b = \mathrm{WMI}(\phi', w' \mid \underbrace{\mathbf{x} \cup \{x_b\}}_{=\mathbf{x}'}, \emptyset). \tag{13}$$

This proves the equivalence of both WMI encodings (with and without Boolean variables)[4]. The isomorphism of the primal graphs of ϕ and ϕ' is trivially satisfied as the transformation performed (replacing Boolean variables with real valued variables) does not introduce nor removes any edges in the primal graph, but only replaces Boolean variable b with continuous variable x_b.

5.2. Measure Theoretic Reduction

Now, the immediate problem with the proof of Proposition 24 presented in the work of Zeng and Van den Broeck [16] is that it is limited to the finite case of Boolean variables. Therefore, it is not clear how to extend it to the case of integer valued variables. Contrary to Boolean variables, integers can take an infinite number of values, which in turn means that, following this procedure, we would have to introduce an infinite number of continuous pieces to turn the summation over integers into a Riemann integration over reals (in the Boolean case we only needed two such pieces, cf. Equation 12). Fortunately, in the light of our results from the preceding sections, we can re-formulate Proposition 24 such that we cover integers as well.

[4] Similar to Zeng and Van den Broeck [16], we limited the discussion here to a single Boolean variable for simplicity of exposition. The argument, however, holds as well for multiple Boolean variables as the elimination of Boolean variables can be performed in succession.

Proposition 26. *For each problem* $\mathrm{WMI}(\phi, w \mid \mathbf{x}, \mathbf{z}, \mathbf{b})$ *(over reals* $\mathbf{x}$*, integers* $\mathbf{z}$ *and Booleans* $\mathbf{b}$*) there exists an equivalent problem* $\mathrm{L{-}WMI}(\phi', w' \mid \mathbf{x}', \mathbf{z}', \mathbf{b}')$ *such that*

$$\mathrm{WMI}(\phi, w \mid \mathbf{x}, \mathbf{z}, \mathbf{b}) = \mathrm{L{-}WMI}(\phi', w' \mid \mathbf{x}', \mathbf{z}', \mathbf{b}')$$

and the primal graphs of ϕ *and* ϕ' *are isomorphic.*

Proof. The proof for the validity of the equality in the proposition follows trivially from Theorem 16 and the observation made in Equation 5. Theorem 16 also gives us $\phi = \phi'$, from which we trivially deduce that the primal graphs of ϕ and ϕ' are isomorphic. □

By simply replacing the Riemann integrations with Lebesgue integrations, the computational complexity discussion in [16, 17, 20] can now be extended to problems where also integer-valued variables are present.

6. Conclusion

WMI is an essential framework for solving probabilistic inference problems in discrete-continuous domains. In this paper we present a measure theoretic formulation of WMI using Lebesgue integration. Consequently, we have ensured conditions for the uniform treatment of problems in Boolean, discrete, continuous domains, and mixtures thereof, which has always been a challenge using classical (Riemannian) theory of integration. Moreover, we have provided clear terminology and precise notation based in measure theory for WMI, putting WMI on steady-state theoretical footing. We see two direct potential benefits of measure theoretic WMI. First, a measure theoretic formulation of WMI might enable novel encoding schemes of WMI problems, which in turn leads to problems being solved more efficiently. The benefits of encodings based on measure theory has already been demonstrated for weighted model counting problems [33] and we stipulate that analog techniques can be adapted for solving WMI problems more efficiently. Secondly, recent advances in using Lebesgue integration for solving integration and related problems have demonstrated potential [34] and could be harnessed in the WMI context, as well. Furthermore, the well-behavedness of Lebesgue integration, with regards to limiting processes, can find its use inside probabilistic inference with potentially infinite number of variables. This is in concordance with a current trend in probabilistic programming research, where an increasing number of papers discuss probabilistic programming from a measure theoretic perspective [35, 36, 31].

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Appendix A. Background on Measure Theory

The theory presented here is taken from [32]. The reader not familiar with measure theory is encouraged to read this introduction for background and motivation, as well as the relationship to classical Riemann integration.

Definition A27 (σ-algebra)**.** Let X be an arbitrary set. A collection $\mathcal{A}$ of subsets of X is a *σ-algebra* on X if:

- $X \in \mathcal{A}$,
- for each set A that belongs to $\mathcal{A}$, its complement A^c belongs to $\mathcal{A}$,
- for each infinite sequence $\{ A_i \}$ of sets that belong to $\mathcal{A}$, set $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathcal{A}$

The pair $(X, \mathcal{A})$ is referred to as *measurable space*. Any set $A \in \mathcal{A}$ is said to be *measurable*.

It is easy to see that the intersection of two σ-algebras is again a σ-algebra. Hence, we can define the smallest σ-algebra which contains the given subsets; it is called the σ-algebra *generated* by these subsets. Now we can introduce an important σ-algebra on the set $\mathbb{R}^N$.

Definition A28 (Borel σ-algebra)**.** The σ-algebra generated by the collection of all rectangles in $\mathbb{R}^N$ that have the form

$$\{ (x_1, \ldots, x_N) \mid a_i < x_i \le b_i, \text{for } i = 1, \ldots, N \}$$

is called *Borel σ-algebra* and is denoted with $\mathcal{B}(\mathbb{R}^N)$.

A function μ from a σ-algebra $\mathcal{A}$ to $[0, +\infty]$ is said to be *countably additive* if it satisfies

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for each infinite sequence $\{ A_i \}$ of disjoint sets from $\mathcal{A}$.

Definition A29. Let $\mathcal{A}$ be a σ-algebra on the set X. The function $\mu \colon \mathcal{A} \to [0, +\infty]$ is a *measure* on $\mathcal{A}$ if $\mu(\emptyset) = 0$ and μ is countably additive. The triple $(X, \mathcal{A}, \mu)$ is said to be a *measure space*.

We now introduce two important measures used in this paper.

Definition A30 (Counting measure)**.** Let X be an arbitrary set, and $\mathcal{P}(X)$ the set of all subsets of X (*power* set). Then $\mathcal{P}(X)$ is trivially a σ-algebra on X. Define a function $\mu \colon \mathcal{P}(X) \to [0, +\infty]$ by letting $\mu(A)$ be n if A is a finite set with exactly n elements, and $+\infty$ otherwise. Then μ is a measure on $\mathcal{P}(X)$ called *counting measure* on $(X, \mathcal{P}(X))$.

Definition A31 (Lebesgue measure)**.** It is possible to construct a function $\lambda^N \colon \mathcal{B}(\mathbb{R}^N) \to [0, +\infty]$ which assigns to each rectangle $R = \{ (x_1, \ldots, x_N) \mid a_i < x_i \le b_i, \text{for } i = 1, \ldots, N \}$ its volume, i.e. $\lambda^N(R) = \prod_{i=1}^{N} (b_i - a_i)$. Then extending λ^N to any set from $\mathcal{B}(\mathbb{R}^N)$ is accomplished by using countable additivity. The function λ^N is a measure on $\mathcal{B}(\mathbb{R}^N)$ and is known as the *Lebesgue measure* on $\mathbb{R}^N$. Sets from $\mathcal{B}(\mathbb{R}^N)$ are in this context often called *Lebesgue measurable sets*.

Let μ be a measure on a measurable space $(X, \mathcal{A})$. Then μ is a *finite measure* if $\mu(X) < +\infty$ and is a *σ-finite measure* if X is the union of a sequence $A_1, A_2, \ldots$ of sets that belong to $\mathcal{A}$ and satisfy $\mu(A_i) < +\infty$ for each $i = 1, 2, \ldots$.

Definition A32 (Measurable function)**.** Let $(X, \mathcal{A})$ be measurable space. The function $f\colon X \to [-\infty, +\infty]$ is said to be *measurable with respect to* $\mathcal{A}$ if for each real number t the set $\{x \in X \mid f(x) \leq t\}$ belongs to $\mathcal{A}$. In the case of $X = \mathbb{R}^N$, a function that is measurable with respect to $\mathcal{B}(\mathbb{R}^N)$ is called *Borel measurable*.

Example A33. There are some familiar measurable functions. For instance, any measurable set $B \in \mathcal{A}$ gives rise to a measurable function. Namely, its characteristic function $\mathbb{1}_B$ is measurable with respect to $\mathcal{A}$, as both B and its complement B^c belong to σ-algebra $\mathcal{A}$. On the other end of spectrum, any continuous function $f\colon \mathbb{R}^N \to \mathbb{R}$ is Borel measurable ([32, Examples 2.1.2 (a)]).

Definition A34 (Simple function)**.** Let $(X, \mathcal{A})$ be a measurable space. Function $f\colon X \to [-\infty, +\infty]$ is called *simple function* if it has only finitely many different values.

Let $a_1, a_2, \ldots, a_n$ be all distinct values of simple function f on measurable space $(X, \mathcal{A})$. Then f can be written as $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, where $A_i = \{\, x \in X \mid f(x) = a_i \,\}$. Function f is $\mathcal{A}$-measurable if and only if $A_i \in \mathcal{A}$ for all $i = 1, 2, \ldots, n$. Simple functions are instrumental in the construction of Lebesgue integral. Their integral is easy to compute, while the next proposition shows that they can approximate any measurable function.

Proposition A35. *Let $(X, \mathcal{A})$ be a measurable space, and let f be a $[0, +\infty]$-valued measurable function on X. Then there is a sequence $\{\, f_n \,\}$ of simple $[0, +\infty)$-valued measurable functions on X that satisfy $f_1(x) \leq f_2(x) \leq \cdots$ and $f(x) = \lim_n f_n(x)$ at each $x \in X$.*

Proof. See [32, Proposition 2.1.8]. □

The construction of integrals takes place in three stages. First, we define an integral of positive simple functions. Using Proposition A35, the definition is then extended to any positive measurable function, and finally extended to the subset of all measurable functions. We denote with f^+ the function $f^+(x) = \max\{\, 0, f(x) \,\}$, i.e. the positive part of function f, and analogously with f^- the function $f^-(x) = -\min\{\, 0, f(x) \,\}$. The function f can now be written as $f = f^+ - f^-$.

Definition A36 (Integral)**.** Let μ be a measure on $(X, \mathcal{A})$. If f is a positive simple function given by $f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$, where $a_1, a_2, \ldots, a_n$ are non-negative real numbers and $A_i, A_2, \ldots, A_n$ are disjoint subsets of X that belong to $\mathcal{A}$, then $\int f \, d\mu$, *the integral of f with respect to μ*, is defined to be $\sum_{i=1}^{n} a_i \, \mu(A_i)$.

For an arbitrary $[0, +\infty]$-valued $\mathcal{A}$-measurable function on X we define its integral as

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu \mid g \text{ is a simple positive function and } g \leq f \right\}.$$

Here we use the shorthand $g \leq f$ to denote that $g(x) \leq f(x)$ holds for all $x \in X$.

Finally, let f be any $[-\infty, +\infty]$-valued $\mathcal{A}$-measurable function on X. If both $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite, then f is called μ-integrable and its integral is defined by

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

Suppose that $f\colon X \to [-\infty, +\infty]$ is $\mathcal{A}$-measurable and that $A \in \mathcal{A}$. Then f is *integrable over* the set A if the function $f \cdot \mathbb{1}_A$ is integrable. In this case $\int_A f \, d\mu$, the integral of f over A, is defined to be $\int f \, \mathbb{1}_A \, d\mu$.

In the case of $X = \mathbb{R}^N$ and $\mu = \lambda^N$, above integral is often referred to as the Lebesgue integral. The Lebesgue integral satisfies all usual basic properties of the Riemann integral, for example linearity and monotonicity.

Proposition A37. *Let $(X, \mathcal{A}, \mu)$ be a measure space, let f and g be measurable functions on X, and let α be any nonnegative real number. Then:*

(a) $\int \alpha f \, d\mu = \alpha \int f \, d\mu$,
(b) $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$, *and*
(c) if $f \le g$ *then* $\int f \, d\mu \le \int g \, d\mu$.

Proof. See [32, Proposition 2.3.4]. □

Importantly, Lebesgue integral equals the Riemman integral for any Riemman integrable function:

Theorem A38. *Let $[a, b]$ be a closed bounded interval, and let f be a bounded real-valued function on $[a, b]$. If f is Riemann integrable, then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.*

Proof. See [32, Theorem 2.5.4 (b)]. □

However, there are functions which are Lebesgue integrable, but not Riemann integrable.

Definition A39. Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that property P holds *μ-almost everywhere* on X or for *μ-almost every $x \in X$ (μ-a.e.)* if there is a set $N \in \mathcal{A}$ such that P holds for every $x \in X \setminus N$ and $\mu(N) = 0$. We omit the mention of measure μ, when it is clear from context.

Lemma A40. *Let $f\colon \mathbb{R}^N \to [-\infty, +\infty]$ be Lebesgue integrable function. Then $\int |f| = 0$ if and only if $f = 0$ almost everywhere.*

Proof. See [32, Corollary 2.3.12]. □

Now we turn to the construction of **product measures**, which combine two measure spaces. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces, and let $X \times Y$ be the Cartesian product of the sets X and Y. A subset of $X \times Y$ is a rectangle with measurable sides if it has the form $A \times B$ for some A in $\mathcal{A}$ and some B in $\mathcal{B}$. The σ-algebra on $X \times Y$ generated by the collection of all rectangles with measurable sides is called the product of the σ-algebras $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \times \mathcal{B}$.

Let E be a subset of $X \times Y$. Then for each $x \in X$ and each $y \in Y$ the *sections* E_x and E^y are subsets of Y and X, respectively, given by $E_x = \{ z \in Y \mid (x, z) \in E \}$ and $E^y = \{ z \in X \mid (z, y) \in E \}$. If f is a function on $X \times Y$, then the *sections* f_x and f^y are functions on Y and X, respectively, given by $f_x(z) = f(x, z)$ and $f^y(z) = f(z, y)$.

Theorem A41 (Product measure)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then there is a* ***unique measure*** *$\mu \times \nu$ on the σ-algebra $A \times B$ such that*

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

holds for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Furthermore, the measure under $\mu \times \nu$ of an arbitrary set E in $A \times B$ is given by

$$(\mu \times \nu)(E) = \int_X \nu(E_x) \, d\mu = \int_Y \mu(E^y) \, d\nu.$$

The measure $\mu \times \nu$ is called the product measure of μ and ν.

Proof. See [32, Theorem 5.1.4]. □

Intgrals with respect to product measure can now be evaluated using Tonelli's theorem, a special case of Fubini's theorem.

Theorem A42 (Tonelli's theorem). *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, and let $f\colon X \times Y \to [0, +\infty]$ be $(\mathcal{A} \times \mathcal{B})$-measurable. Then*

(a) the function $x \mapsto \int_Y f_x \, d\nu$ is $\mathcal{A}$-measurable and the function $y \mapsto \int_X f^y \, d\mu$ is $\mathcal{B}$-measurable, and

(b) f satisfies

$$\int_{X\times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f_x \, d\nu \right) d\mu = \int_Y \left(\int_X f^y \, d\mu \right) d\nu.$$

Proof. See [32, Theorem 5.2.1]. □

Probability theory is naturally expressed in terms of Lebesgue integration.

Definition A43. A *probability space* is a measure space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{P}(\Omega) = 1$. A measure $\mathbb{P}$ is called a *probability*.

Let now $(X, \mathcal{A}, \mu)$ be a measure space. Suppose that f is a non-negative μ-measurable function on X such that $\int_X f \, d\mu = 1$. Then the function $\mathbb{P}\colon \mathcal{A} \to [0, 1]$ given with $\mathbb{P}(A) = \int_A f \, d\mu$, for every $A \in \mathcal{A}$, defines a probability on the measurable space $(X, \mathcal{A})$. The function f is called the *probability density function (PDF)* of probability $\mathbb{P}$.

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